CS103
Winter 2025

Lecture 07:

Functions

Part 2 of 2

Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

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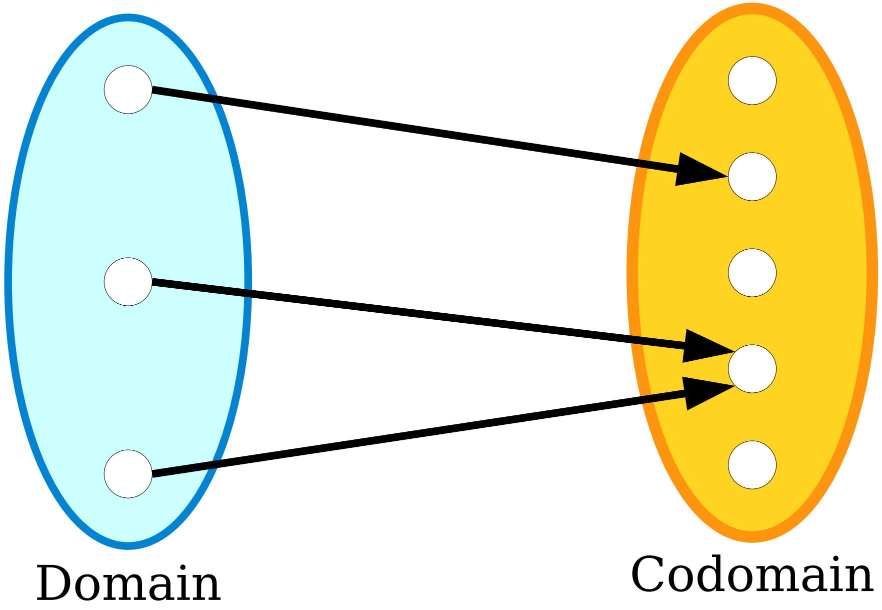
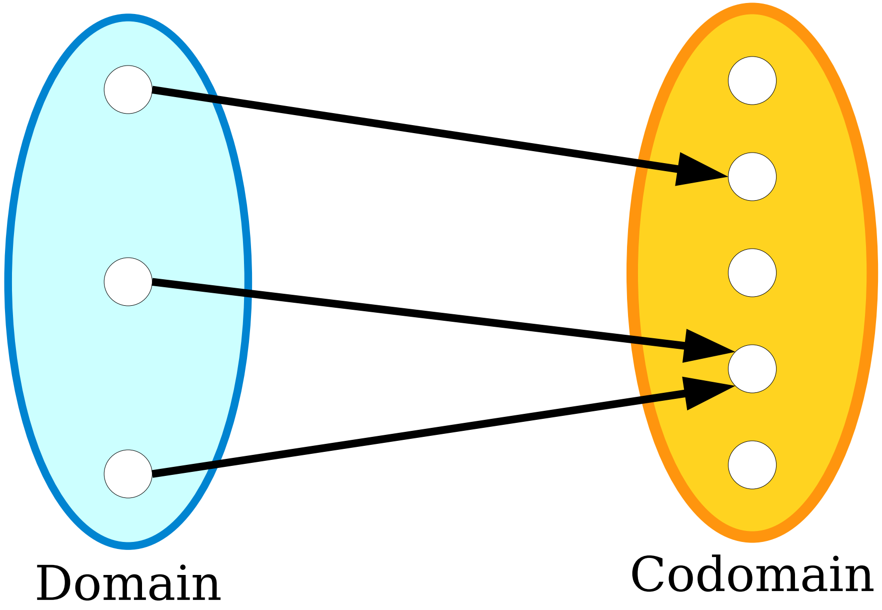


Recap from Last Time

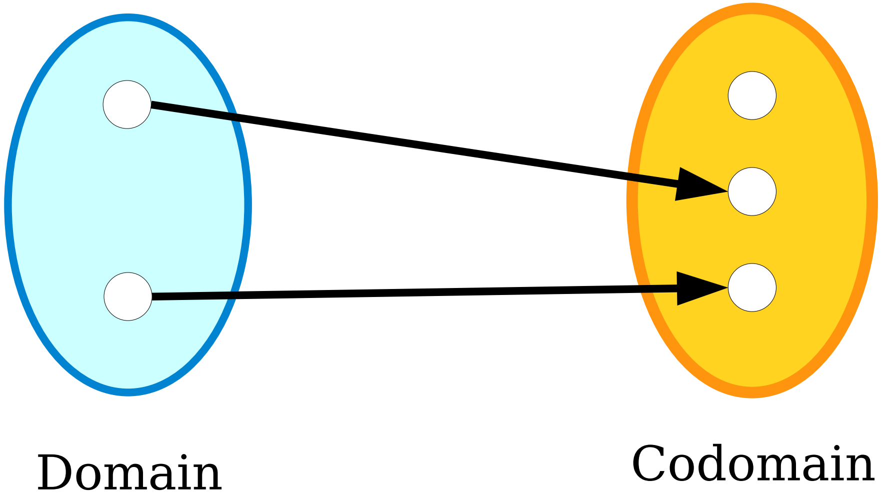


Is it a function? **Yes!**

Is it an injection? **No.**

Is it a surjection? **No.**

Recap from Last Time



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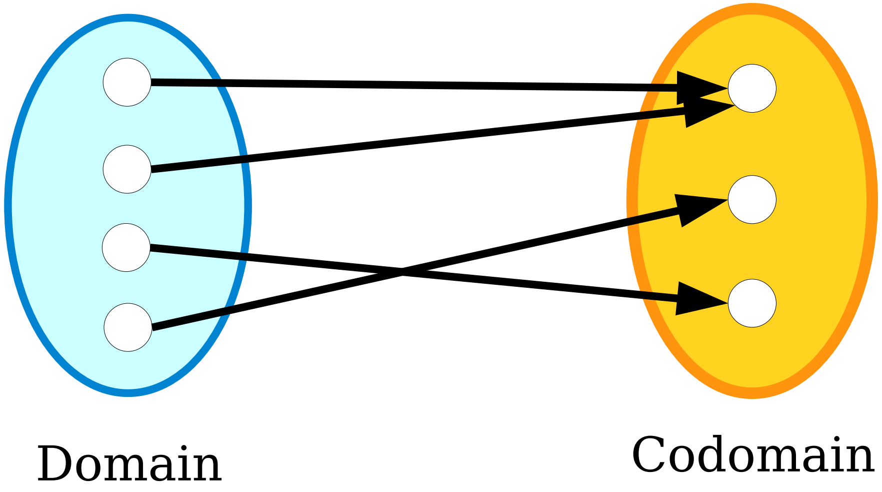


Is it a function? **Yes!**

Is it an injection? **No.**

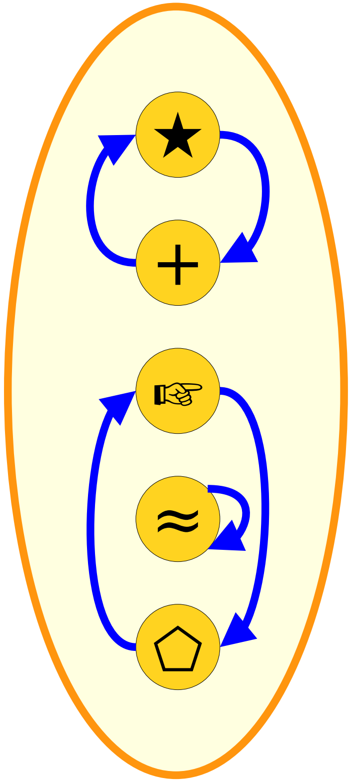
Is it a surjection? **Yes!**

Recap from Last Time

Injection: $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$

If the inputs are different, the outputs are different

- Can also define with the contrapositive!



Surjection: $\forall b \in B.\ \exists a \in A.\ f(a) = b$

*"For every possible output,
there's an input that produces it."*

Involution: $\forall x \in A.\ f(f(x)) = x$

*"Applying f twice is equivalent
to not applying f at all."*

		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

New Stuff!

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

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Given the predicates

$Bird(b)$, which says b is a bird;
$Heron(h)$, which says h is a heron; and
$Feathers(x)$, which says x has feathers,

translate the theorem into first-order logic.

Answer at
https://cs103.stanford.edu/pollev

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translate the theorem into first-order logic.

$$\underbrace{(\forall b.\,(\mathit{Bird}(b) \rightarrow \mathit{Feathers}(b)))}_{\text{All birds have feathers}} \rightarrow \underbrace{(\forall h.\,(\mathit{Heron}(h) \rightarrow \mathit{Feathers}(h)))}_{\text{All herons have feathers}}$$

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Proof: Assume that all birds have feathers. We will show that all herons have feathers.

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Which makes more sense as the next step in this proof?

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Consider an arbitrary bird *b*. Since *b* is a bird, *b* has feathers. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds have feathers.
  - Here, we ***proved*** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

  $$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$\underbrace{(\forall b.\ (Bird(b) \to Feathers(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (Heron(h) \to Feathers(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- If we ***assume*** the statement

  $$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, have feathers.

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$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
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$\exists x. A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
    - ***involutions***, functions that undo themselves;
    - ***injections***, functions where different inputs go to different outputs; and
    - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{f \text{ is an involution.}} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{f \text{ is surjective.}}$$

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Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \ \to\ \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

We've said that we need to *prove* this statement. How do we do that?

What do you do to prove
$\forall b \in A$. [something]?

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To ***prove*** that
this is true…

Have the reader pick an arbitrary x. Then prove A is true for that choice of x.

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There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

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Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

Prove this.

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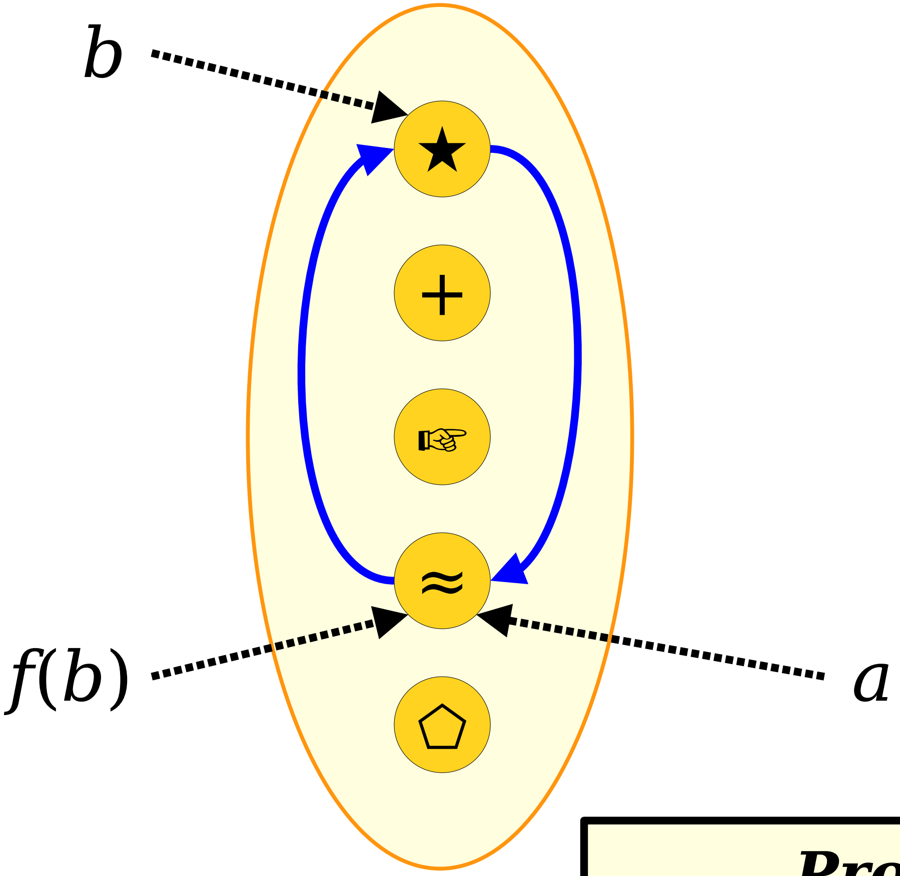

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Proof: Pick any involution $f : A \to A$. We will prove that f is surjective.

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Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$.

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Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Proof Outline

1. Assume f is an involution.
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The Two-Column Proof Organizer

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

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What We're Assuming

$f : A \to A$ is an involution.

$$\forall z \in A.\ f(f(z)) = z.$$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

What We Need to Prove

f is injective.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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$a_2 \in A$

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f is injective.

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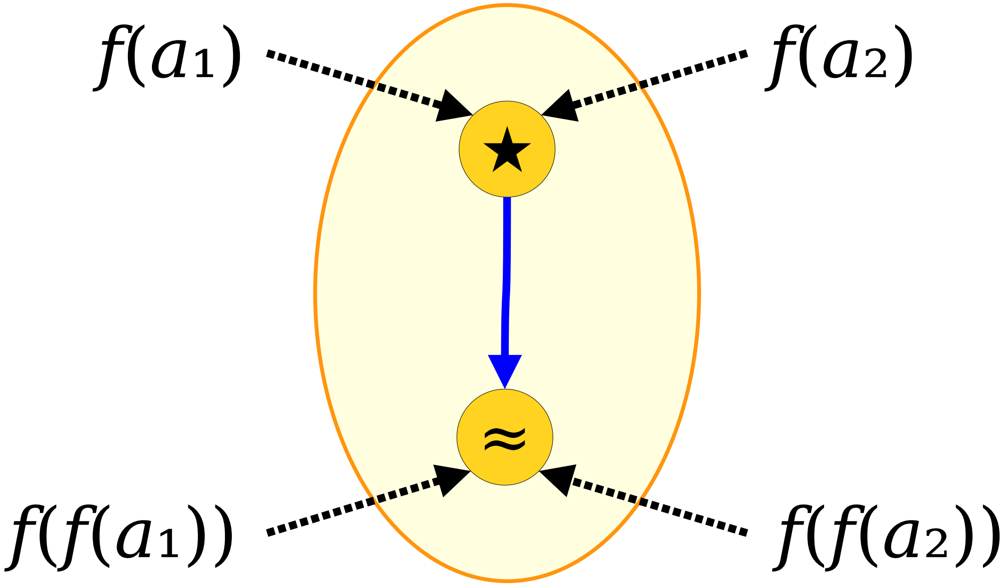

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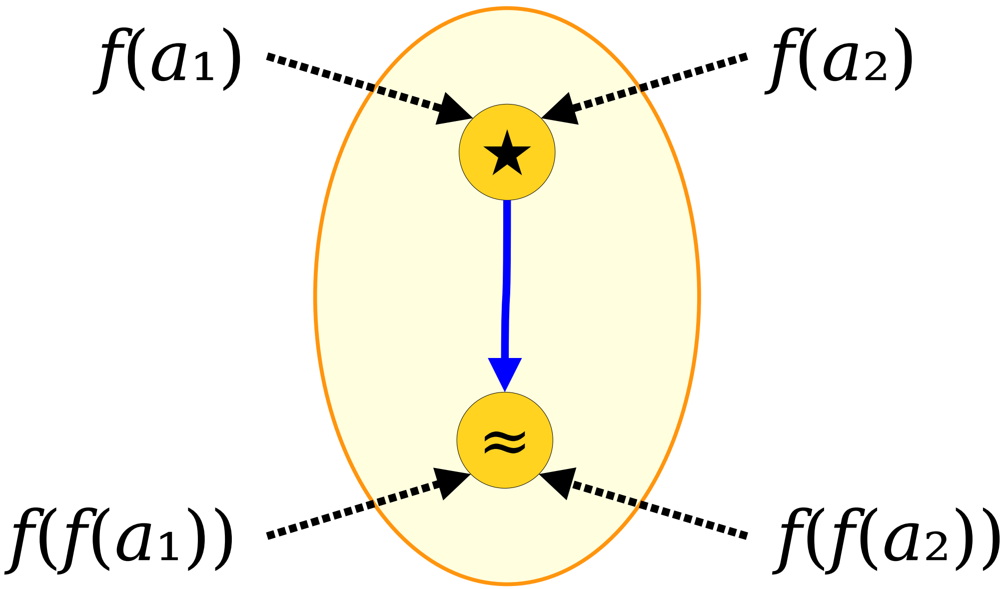

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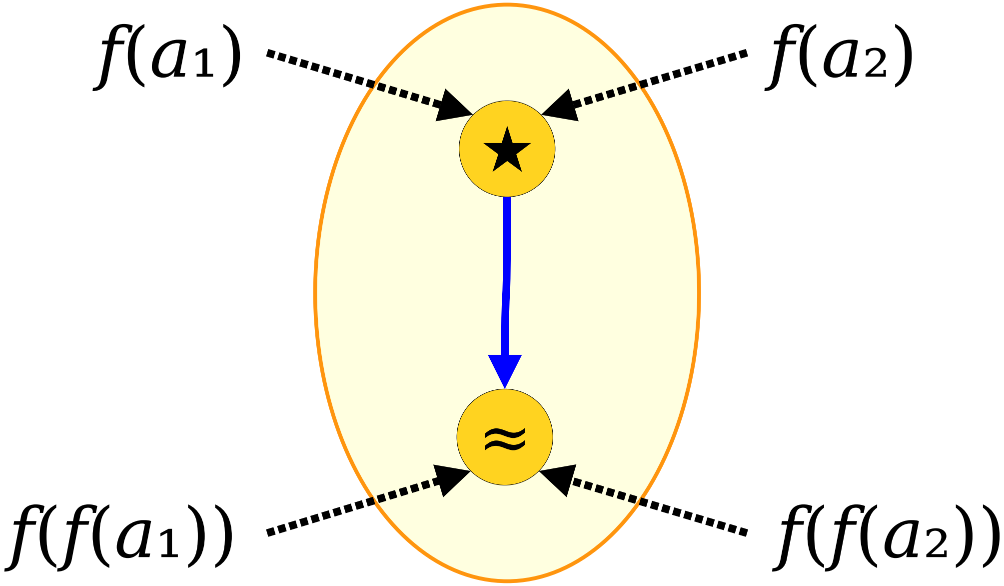

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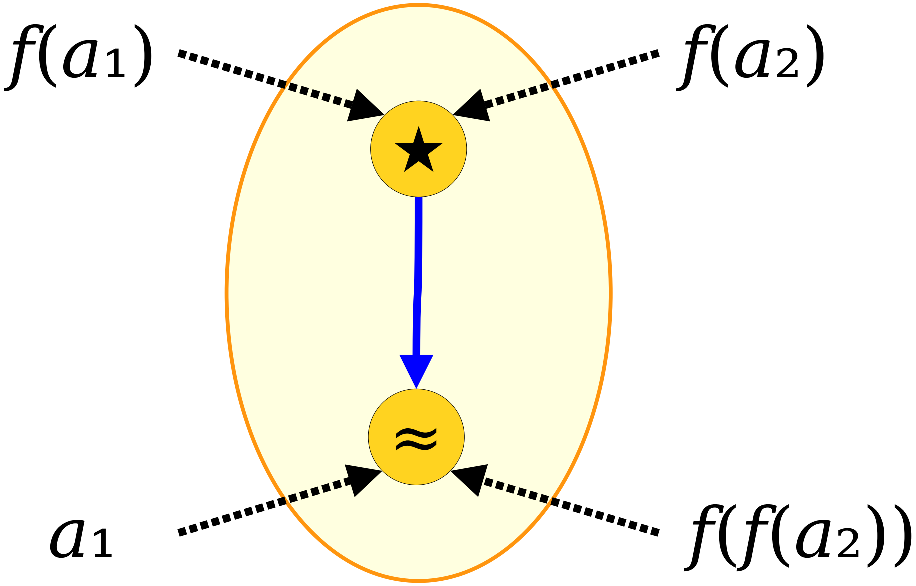

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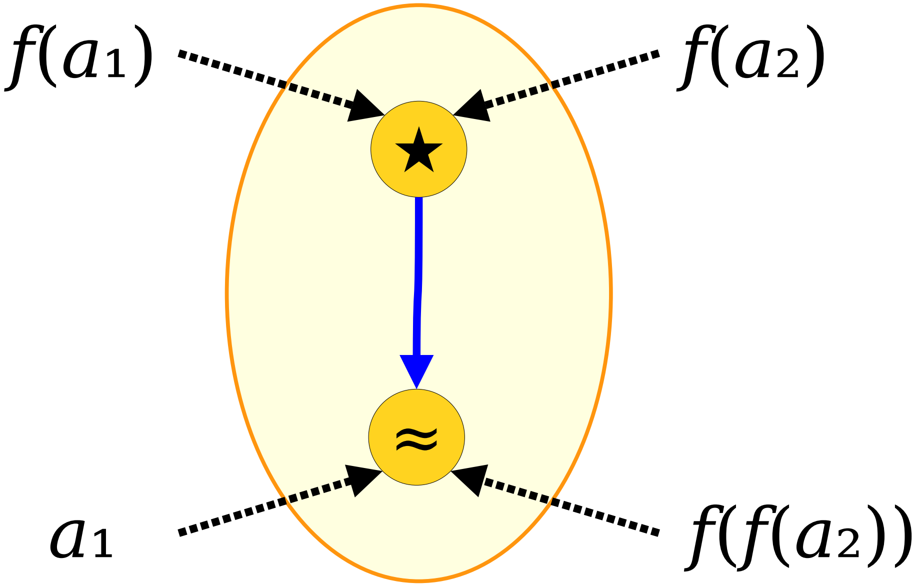

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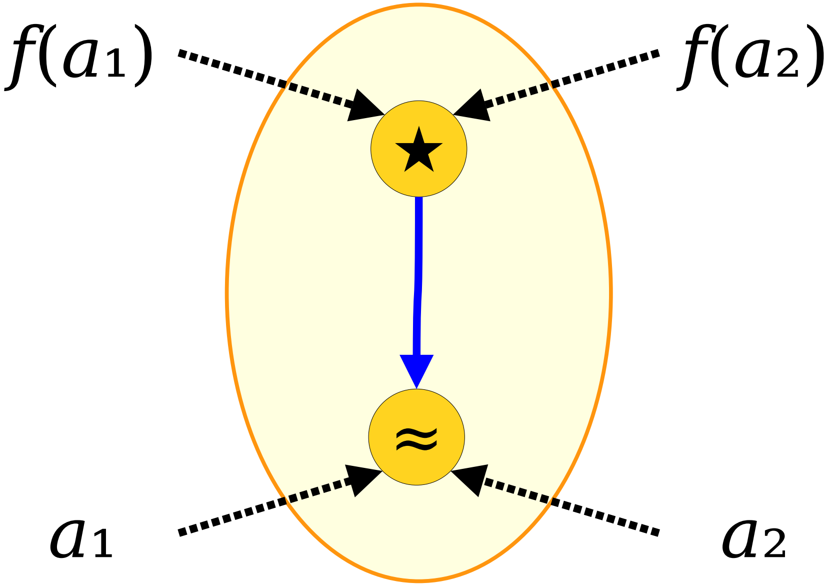

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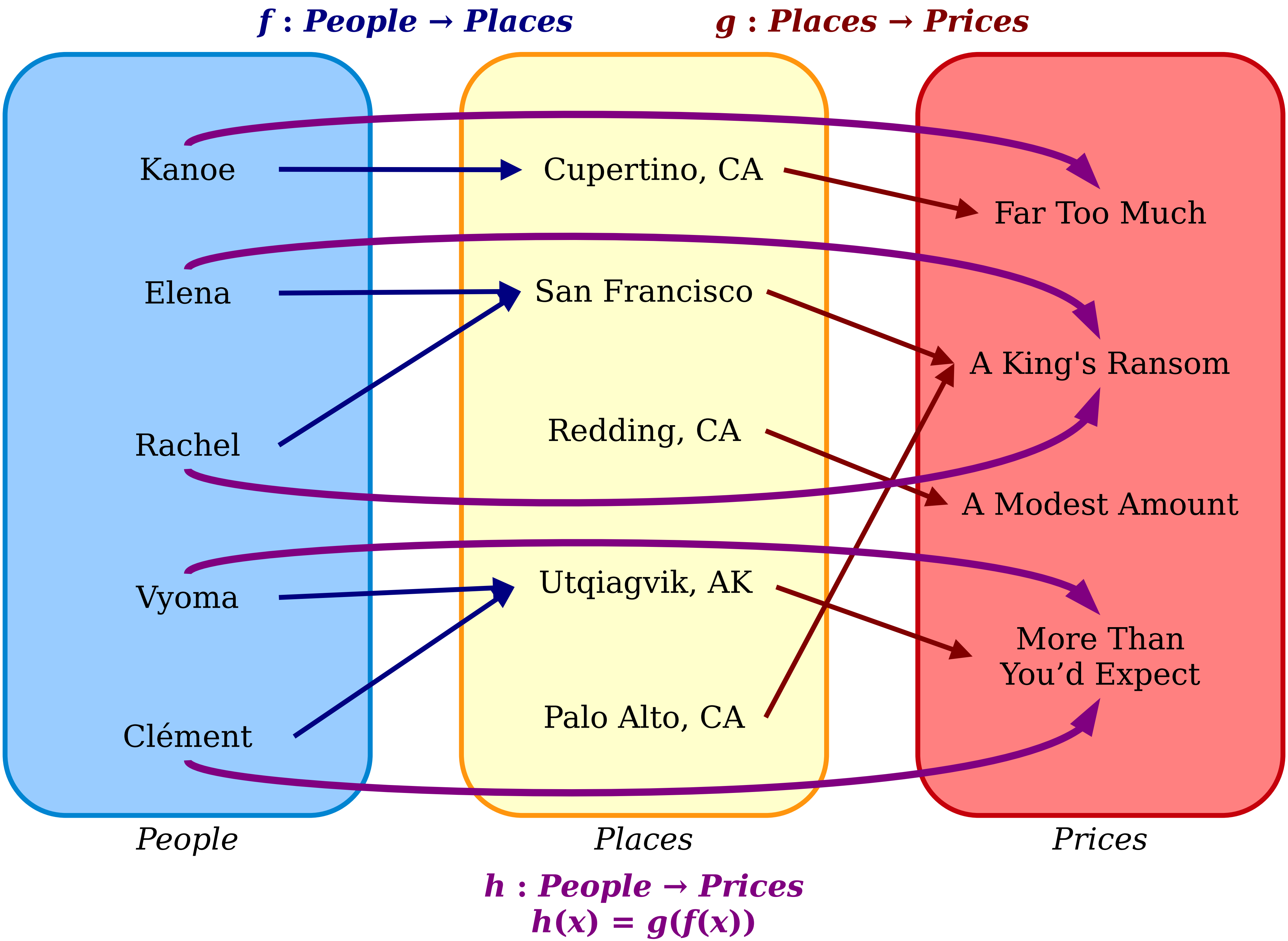
Time-Out for Announcements!

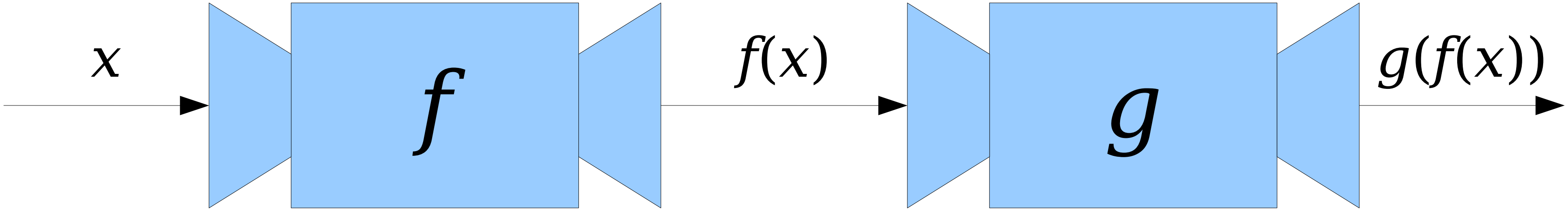
Back to CS103!

Function Composition


f : People → Places
g : Places → Prices
Kanoe
Elena
Rachel
Vyoma
Clément
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You'd Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to \\ f(x) \neq f(y) \\ )$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to \\ g(x) \neq g(y) \\ )$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to \\ (g \circ f)(a_1) \neq (g \circ f)(a_2) \\ )$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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Now we’re looking at
an implication. Let’s
assume the antecedent
and prove the consequent.

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Let's write this out separately and simplify things a bit.

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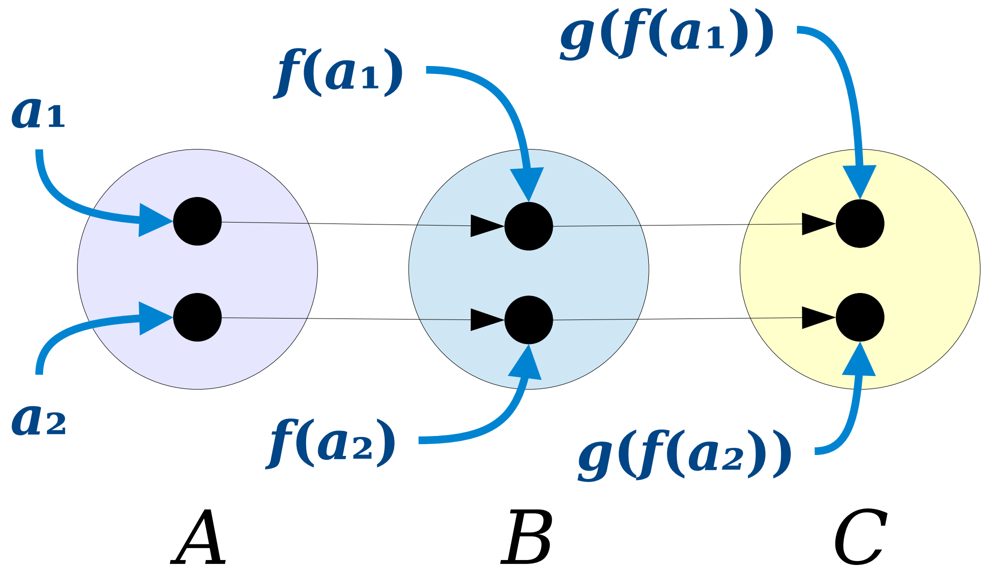
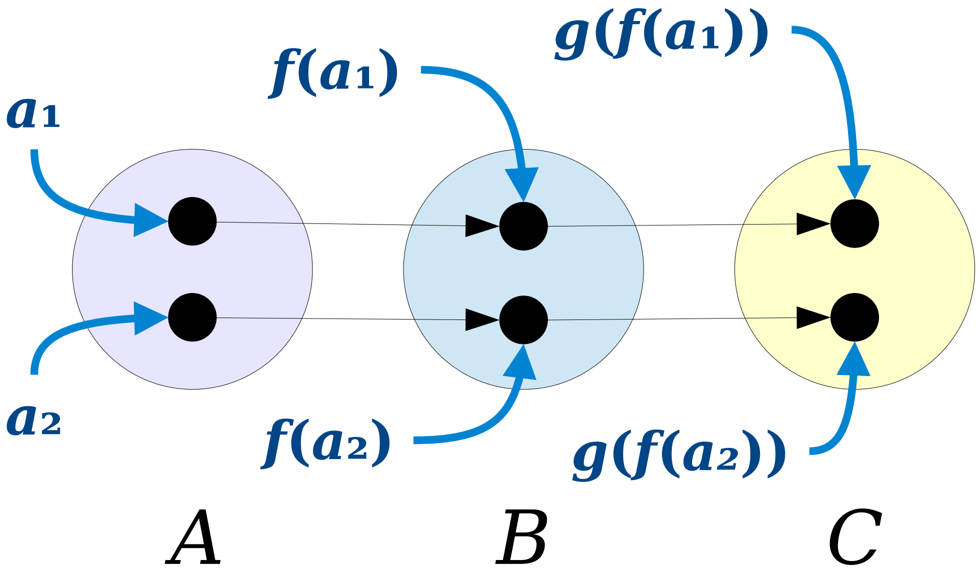


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



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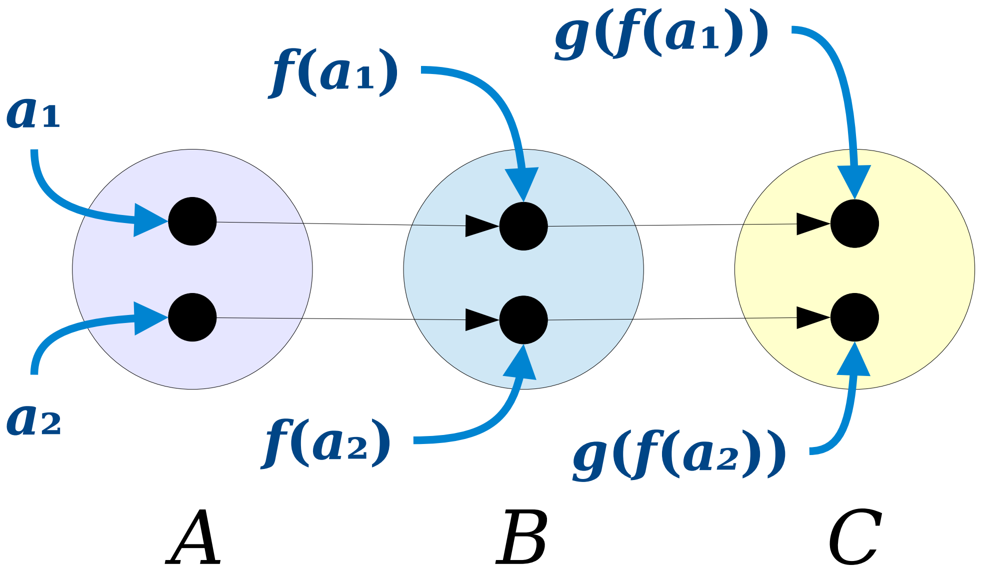

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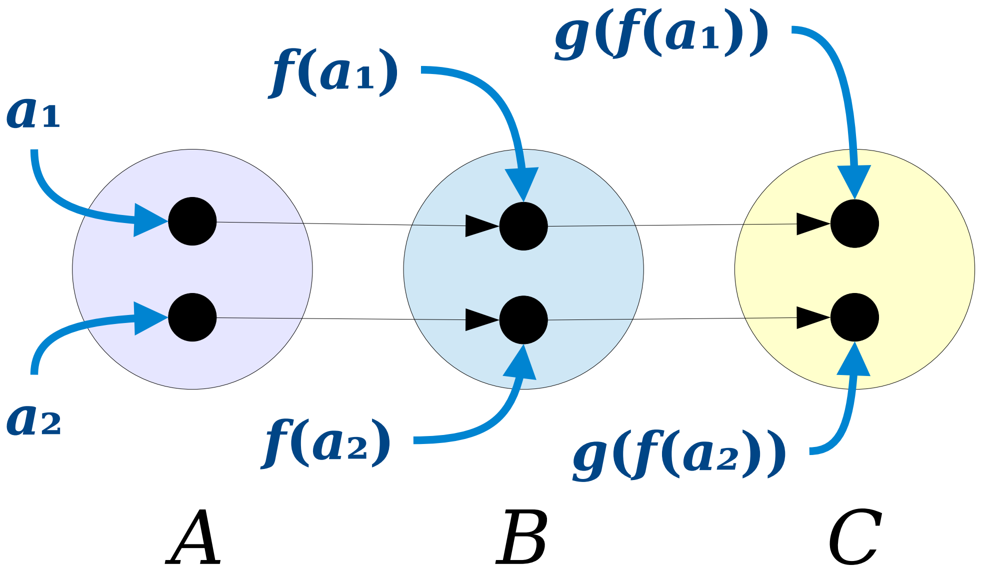

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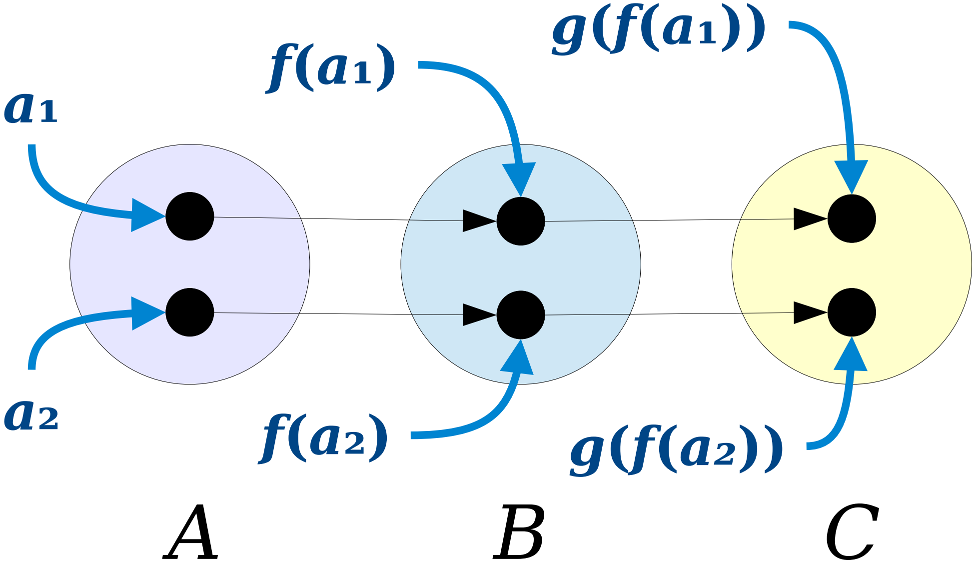

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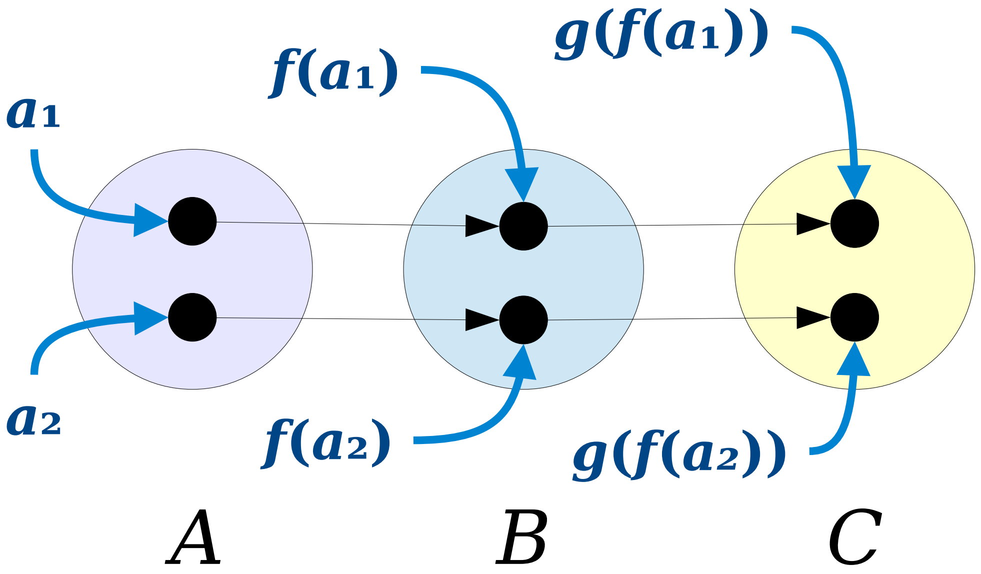

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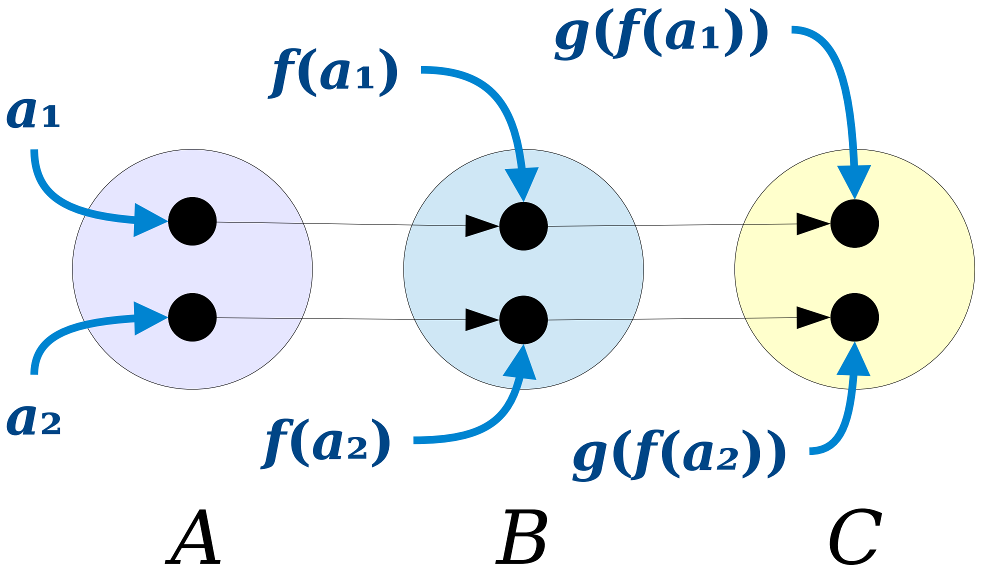

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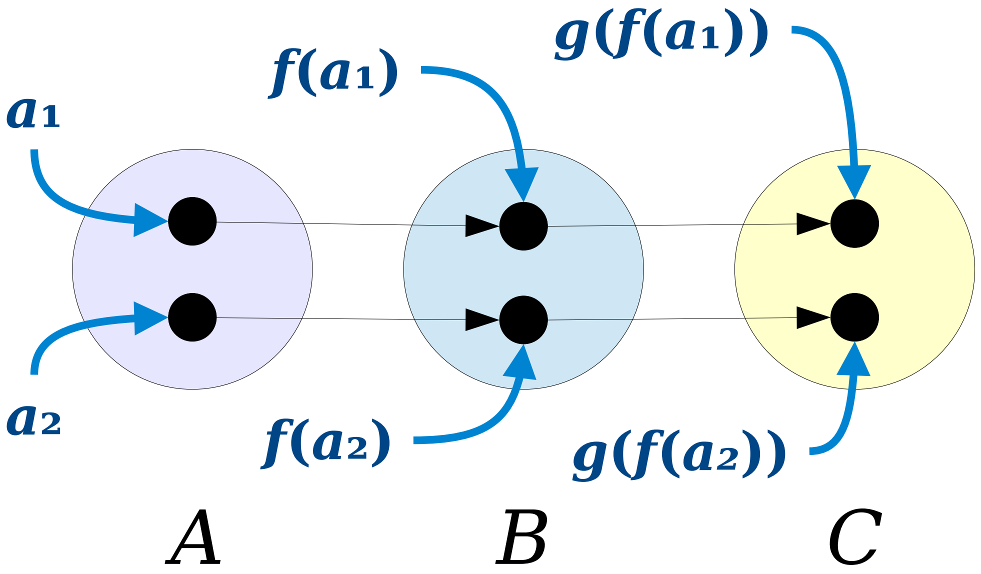

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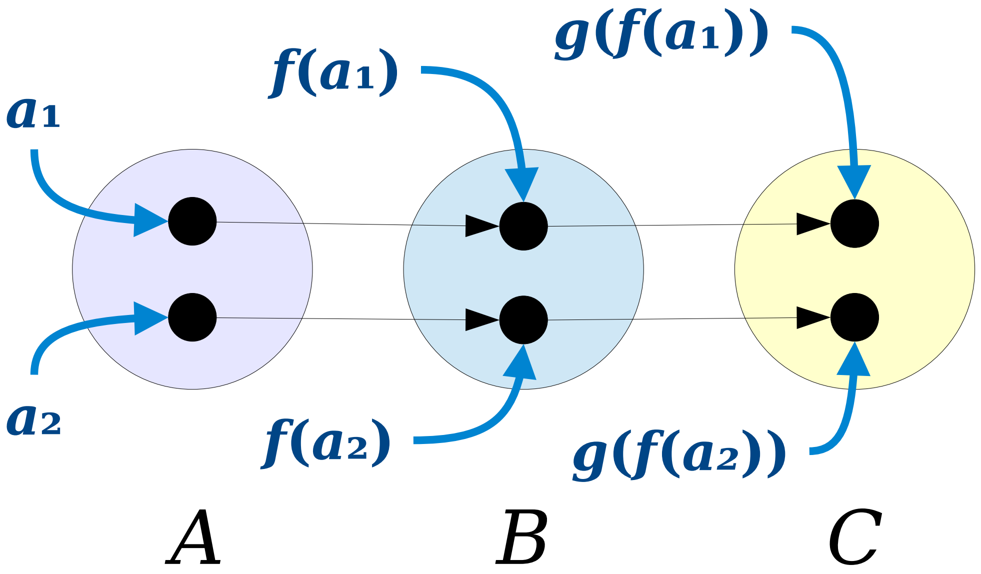
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



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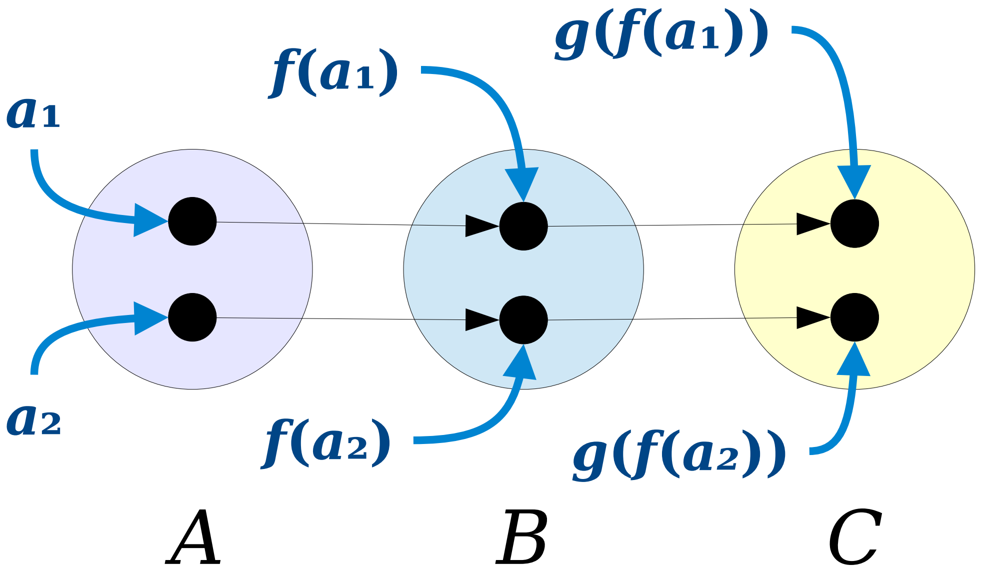
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.



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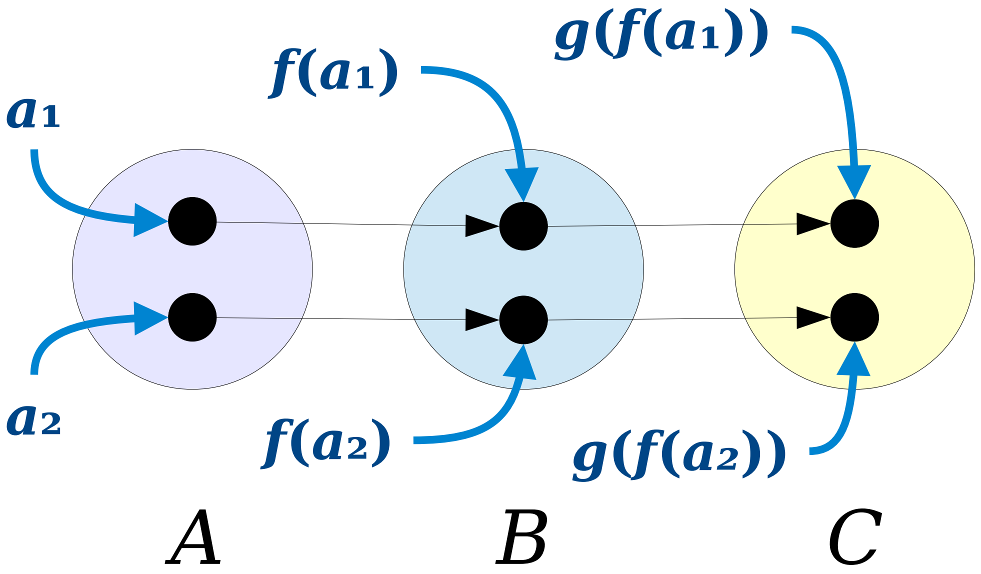

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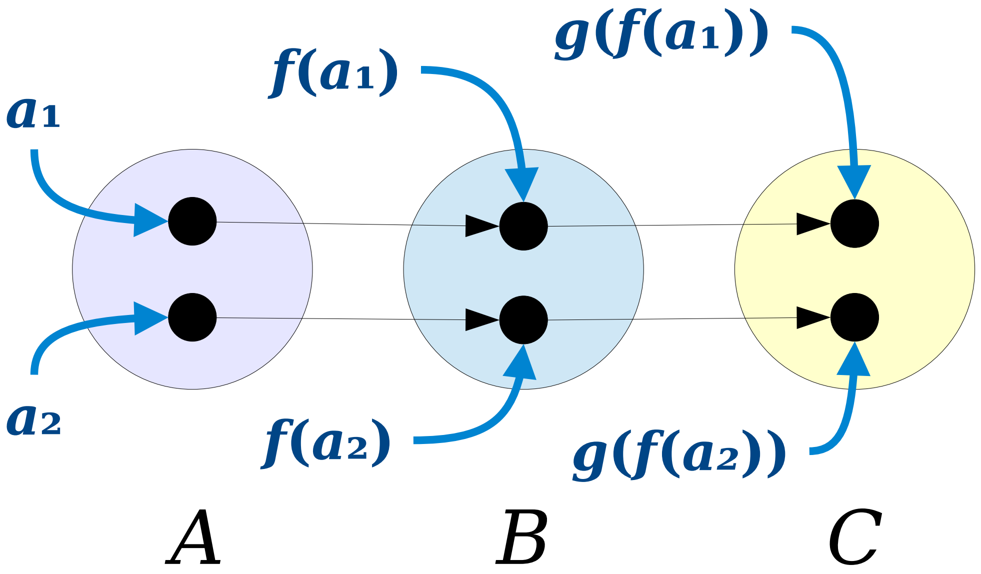
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



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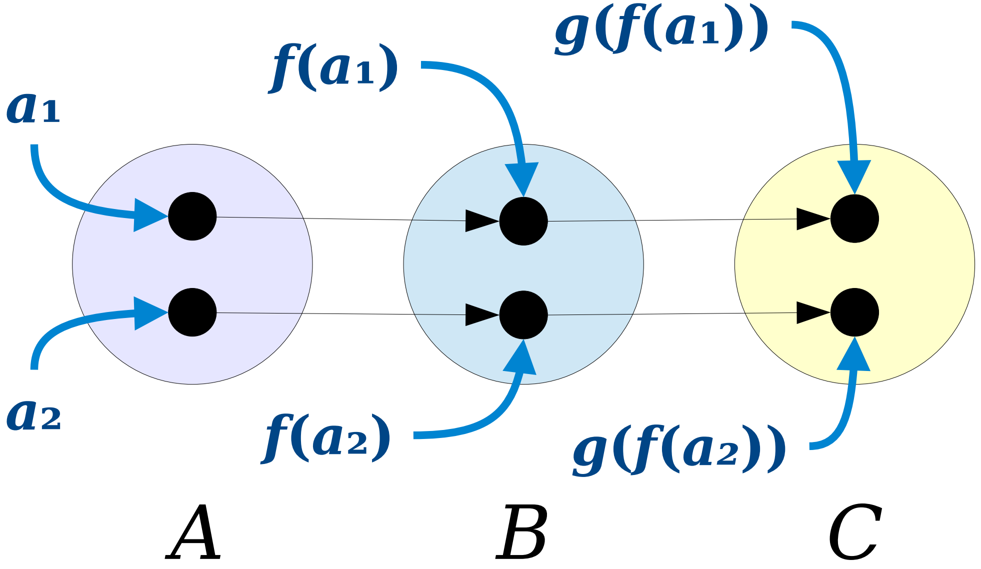
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Proofs on Sets***
  - How to rigorously establish set-theoretic results.

Appendix: Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

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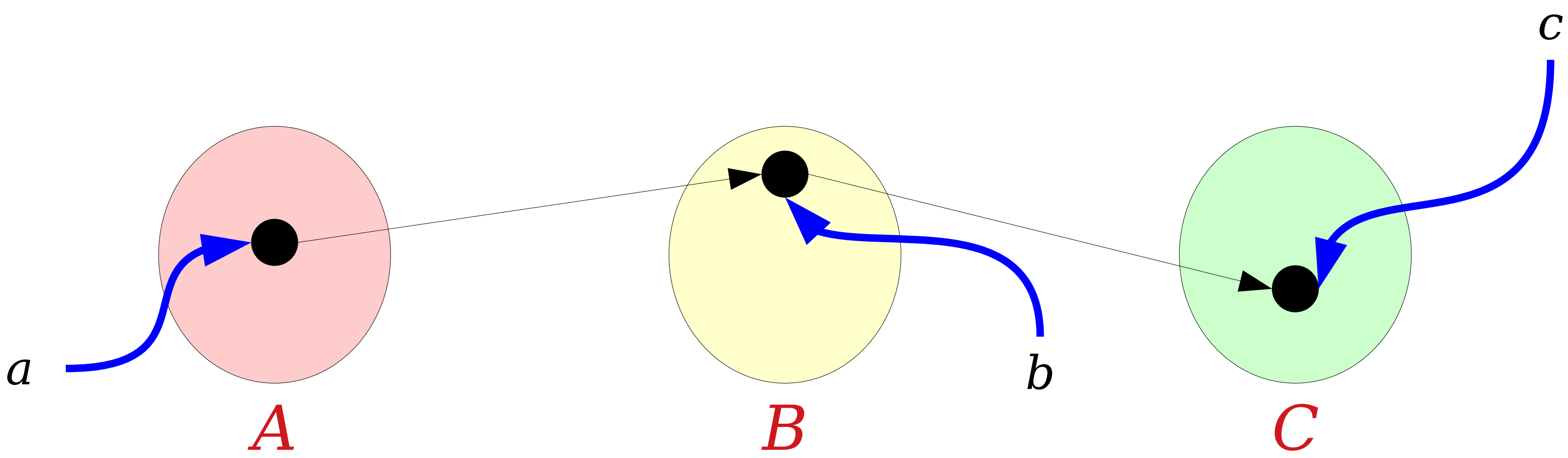

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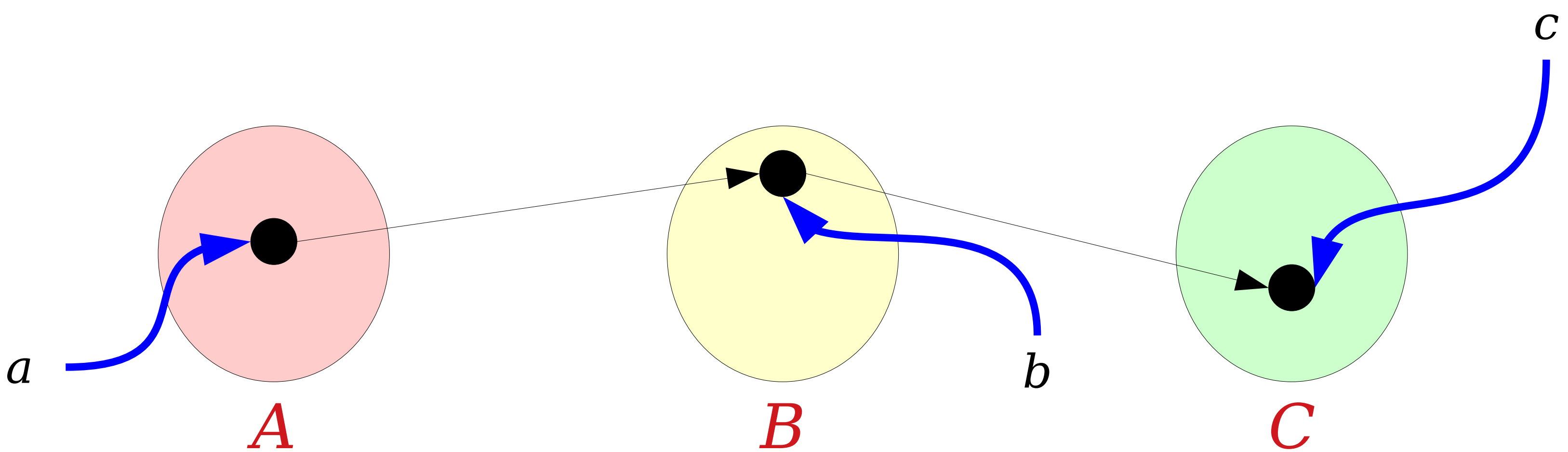
Consider any $c \in C$.



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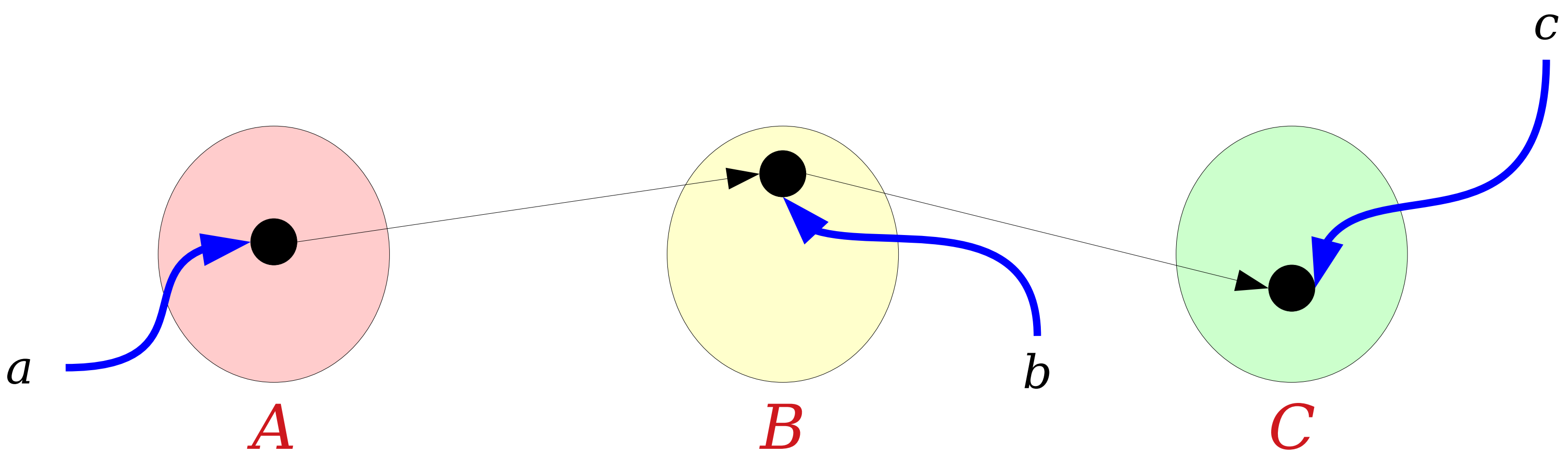

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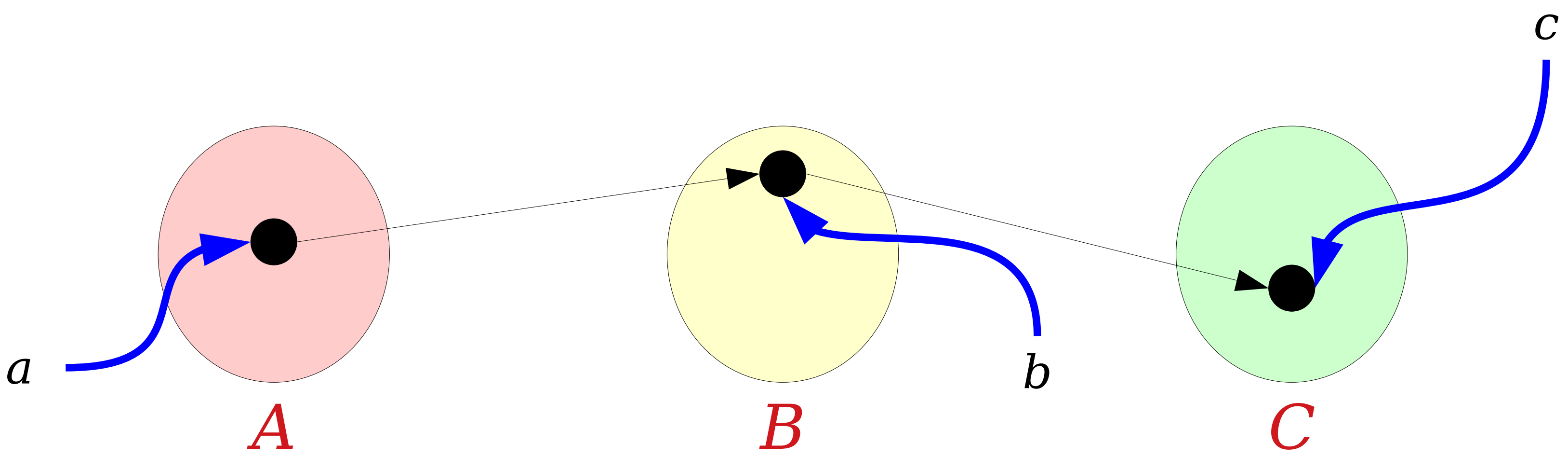

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$$g(f(a)) = g(b) = c,$$

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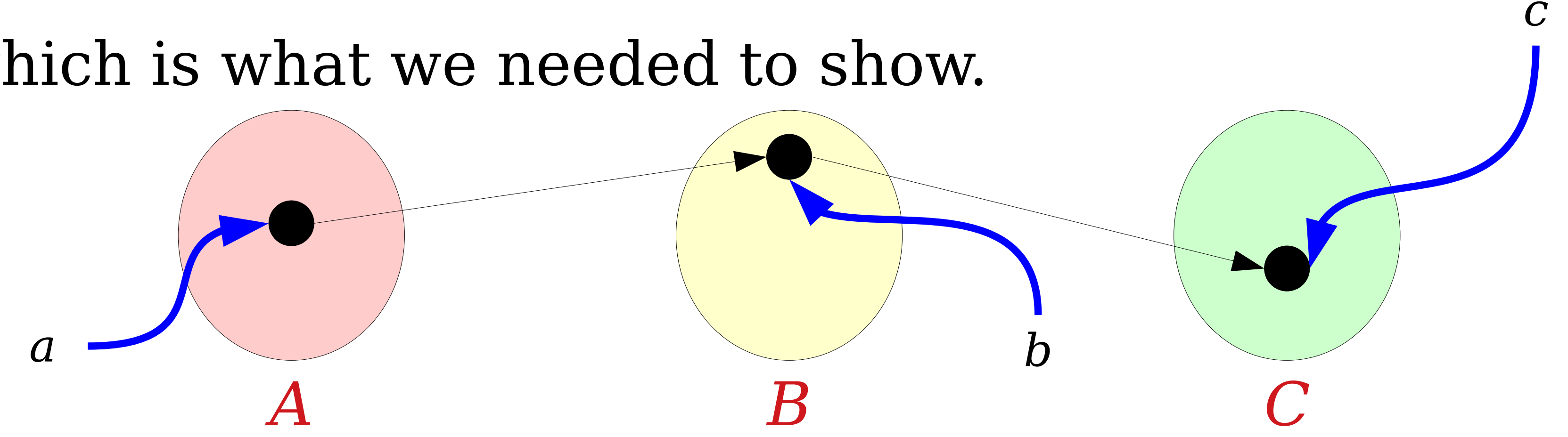

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